TÉCNICAS DE INTEGRAÇÃO

Substituição trigonométrica

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Onde usar a substituição trigonométrica:

Considere uma integral da forma $\int \sqrt{a^2 - x^2}\, dx$. *Se fizermos* $x = a \operatorname{sen} \theta$, *temos:* $\sqrt{a^2 - x^2} = \sqrt{a^2 - a^2 \operatorname{sen}^2 \theta} = \sqrt{a^2(1 - \operatorname{sen}^2 \theta)} = \sqrt{a^2} \cdot \sqrt{1 - \operatorname{sen}^2 \theta} = a \cdot \sqrt{\cos^2 \theta} = a \cdot |cos\theta|$. *Substituições como essa aparecem nas seguintes expressões:*

Expressão	Substituição	Identidade
$\sqrt{a^2 - x^2}$	$x = a \operatorname{sen} \theta,\ -\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$	$1 - \operatorname{sen}^2 \theta = \cos^2 \theta$
$\sqrt{x^2 + a^2}$	$x = a \operatorname{tg} \theta,\ -\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$	$1 + \operatorname{tg}^2 \theta = \sec^2 \theta$
$\sqrt{x^2 - a^2}$	$x = a \sec \theta,\ 0 \leq \theta \leq \frac{\pi}{2}$ ou $\pi \leq \theta \leq \frac{3\pi}{2}$	$\sec^2 \theta - 1 = \operatorname{tg}^2 \theta$

Exemplo 1: Avalie a integral $\int \frac{1}{x^2\sqrt{25 - x^2}}\, dx$:

Se $x = 5 \operatorname{sen} \theta \Rightarrow dx = 5 \cos \theta\, d\theta$ e com isso:

$$\sqrt{25 - x^2} =$$

$$\sqrt{25 - (5 \operatorname{sen} theta)^2} =$$

$$\sqrt{25 - 25 \operatorname{sen}^2 \theta} =$$

$$\sqrt{25(1 - \operatorname{sen}^2 \theta)} =$$

$$\sqrt{25} \cdot \sqrt{1 - \operatorname{sen}^2 \theta} =$$

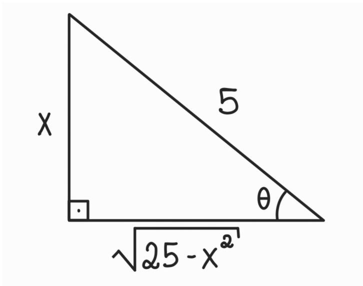
$$5 \cdot \sqrt{\cos^2 \theta} =$$

$$5 \cos \theta$$

Logo:

$$\int \frac{1}{x^2\sqrt{25-x^2}}\ dx =$$
$$\int \frac{1}{(5\,\text{sen}\,\theta)^2 \cdot 5\cos\theta}\ 5\cos\theta\ d\theta =$$
$$\int \frac{1}{25\,\text{sen}^2\,\theta}\ d\theta =$$
$$\frac{1}{25}\int \frac{1}{\text{sen}^2\,\theta}\ d\theta =$$
$$\frac{1}{25}\int \text{cossec}^2\,\theta\ d\theta =$$
$$-\frac{1}{25}\,\text{cotg}\,\theta + C$$

Como $x = 5\,\text{sen}\,\theta \Rightarrow \text{sen}\,\theta = \dfrac{x}{5}$ e $\text{cotg}\,\theta = \dfrac{CA}{CO}$:



$$\text{cotg}\,\theta = \frac{\sqrt{25-x^2}}{x}$$

Então:

$$\int \frac{1}{x^2\sqrt{25-x^2}}\ dx = -\frac{1}{25}\,\text{cotg}\,\theta + C = -\frac{1}{25}\cdot\frac{\sqrt{25-x^2}}{x} + C = -\frac{\sqrt{25-x^2}}{25x} + C$$

Exemplo 2: Avalie a integral: $\int \dfrac{dx}{x\sqrt{x^2+3}}$:

Se $x = \sqrt{3}\,\text{tg}\,\theta \Rightarrow dx = \sqrt{3}\sec^2\theta\ d\theta$, então:

$$\sqrt{x^2+3} =$$
$$\sqrt{(\sqrt{3}\,\text{tg}\,\theta)^2+3} =$$
$$\sqrt{3\,\text{tg}^2\,\theta+3} =$$
$$\sqrt{3(\text{tg}^2\,\theta+1)} =$$
$$\sqrt{3}\cdot\sqrt{\text{tg}^2\,\theta+1} =$$
$$\sqrt{3}\cdot\sqrt{\sec^2\theta} =$$
$$\sqrt{3}\sec\theta$$

Logo:

$$
\begin{aligned}
\int \frac{dx}{x\sqrt{x^2+3}} &= \\
\int \frac{\sqrt{3}\sec^2\theta\ d\theta}{\sqrt{3}\operatorname{tg}\theta\cdot\sqrt{3}\sec\theta} &= \\
\int \frac{\sqrt{3}\sec\theta}{3\cdot\operatorname{tg}\theta}\ d\theta &= \\
\frac{\sqrt{3}}{3}\int \frac{\sec\theta}{\operatorname{tg}\theta}\ d\theta &= \\
\frac{\sqrt{3}}{3}\int \frac{\frac{1}{\cos\theta}}{\frac{\cos\theta}{\operatorname{sen}\theta}}\ d\theta &= \\
\frac{\sqrt{3}}{3}\int \frac{1}{\cos\theta}\cdot\frac{\cos\theta}{\operatorname{sen}\theta}\ d\theta &= \\
\frac{\sqrt{3}}{3}\int \frac{1}{\operatorname{sen}\theta}\ d\theta &= \\
\frac{\sqrt{3}}{3}\int \operatorname{cossec}\theta\ d\theta
\end{aligned}
$$

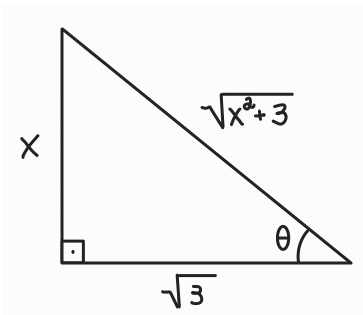
Como:

$$
\begin{aligned}
\frac{d}{d\theta}(-\ln|\operatorname{cossec}\theta+\operatorname{cotg}\theta|) &= \\
\frac{\operatorname{cossec}\theta\cdot\operatorname{cotg}\theta+\operatorname{cossec}^2\theta}{\operatorname{cossec}\theta+\operatorname{cotg}\theta} &= \\
\frac{\operatorname{cossec}\theta\cdot(\operatorname{cotg}\theta+\operatorname{cossec}\theta)}{\operatorname{cotg}\theta+\operatorname{cossec}\theta} &= \\
&\operatorname{cossec}\theta
\end{aligned}
$$

Então:

$$\int \frac{dx}{x\sqrt{x^2+3}} = \frac{\sqrt{3}}{3}\int \operatorname{cossec}\theta\ d\theta = -\frac{\sqrt{3}\ln|\operatorname{cossec}\theta+\operatorname{cotg}\theta|}{3} + C$$

Além disso, obtém-se que:



$$\operatorname{cossec}\theta = \frac{1}{\operatorname{sen}\theta} = \frac{H}{CO} = \frac{\sqrt{x^2+3}}{x}$$

e também:

$$\operatorname{cotg}\theta = \frac{1}{\operatorname{cotg}\theta} = \frac{CA}{CO} = \frac{\sqrt{3}}{x}$$

Por fim:

$$\int \frac{dx}{x\sqrt{x^2+3}} = \frac{\sqrt{3}}{3}\int \operatorname{cossec}\theta\ d\theta = -\frac{\sqrt{3}\ln|\operatorname{cossec}\theta + \operatorname{cotg}\theta|}{3} + C =$$
$$-\frac{\sqrt{3}\ln\left|\dfrac{\sqrt{x^2+3}}{x} + \dfrac{\sqrt{3}}{x}\right|}{3} + C = -\frac{\sqrt{3}\ln\left|\dfrac{\sqrt{x^2+3}+\sqrt{3}}{x}\right|}{3} + C$$

Exemplo 3: Avalie a integral $\int_{\sqrt{2}}^{2} \frac{1}{t^3}\sqrt{t^2-1}\ dt$:
Seja $t = \sec\theta$, então: $dt = \sec\theta \cdot \operatorname{tg}\theta\ d\theta$ e também:

$$\begin{aligned} \sqrt{t^2-1} &= \\ \sqrt{\sec^2\theta - 1} &= \\ \sqrt{\operatorname{tg}^2\theta} &= \\ &\operatorname{tg}\theta \end{aligned}$$

Quando $t = \sqrt{2} \Rightarrow \sec\theta = \sqrt{2} \Rightarrow \frac{1}{\cos\theta} = \sqrt{2} \Rightarrow \cos\theta = \frac{1}{\sqrt{2}} \Rightarrow \cos\theta = \frac{\sqrt{2}}{2} \Rightarrow \theta = \frac{\pi}{4}$
E quando $t = 2 \Rightarrow \sec\theta = 2 \Rightarrow \frac{1}{\cos\theta} = 2 \Rightarrow \cos\theta = \frac{1}{2} \Rightarrow \theta = \frac{\pi}{3}$
Logo:

$$\begin{aligned} \int_{\sqrt{2}}^{2} \frac{\sqrt{t^2-1}}{t^3}\ dt &= \\ \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \frac{\operatorname{tg}\theta}{\sec^3\theta} \cdot \sec\theta\operatorname{tg}\theta\ d\theta &= \\ \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \frac{\operatorname{tg}^2\theta}{\sec^2\theta}\ d\theta &= \\ \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \frac{\sec^2\theta - 1}{\sec^2\theta}\ d\theta &= \\ \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \frac{\sec^2\theta}{\sec^2\theta} - \frac{1}{\sec^2\theta}\ d\theta &= \\ \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} 1 - \cos^2\theta\ d\theta &= \end{aligned}$$

$$\int_{\frac{\pi}{4}}^{\frac{\pi}{3}} 1 - \frac{1+\cos 2\theta}{2}\ d\theta =$$

$$\int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \frac{2}{2} - \frac{1}{2} - \frac{\cos 2\theta}{2}\ d\theta =$$

$$\int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \frac{1}{2} - \frac{\cos 2\theta}{2}\ d\theta =$$

$$\frac{1}{2}\int_{\frac{\pi}{4}}^{\frac{\pi}{3}} 1 - \cos 2\theta\ d\theta =$$

$$\frac{1}{2}\left(\theta - \frac{\operatorname{sen} 2\theta}{2}\right)\Bigg|_{\frac{\pi}{4}}^{\frac{\pi}{3}} =$$

$$\frac{1}{2}\left(\frac{\pi}{3} - \frac{\operatorname{sen}\frac{2\pi}{3}}{2} - \frac{\pi}{4} + \frac{\operatorname{sen}\frac{\pi}{2}}{2}\right) =$$

$$\frac{1}{2}\left(\frac{\pi}{12} - \frac{\sqrt{3}}{4} + \frac{1}{2}\right) =$$

$$\frac{1}{2}\left(\frac{\pi - 3\sqrt{3} + 6}{12}\right) =$$

$$\frac{\pi - 3\sqrt{3} + 6}{24}$$

Referências

[1] GUIDORIZZI, Hamilton Luiz. *Um curso de cálculo: volume 1*. Rio de Janeiro. LTC–Livros Técnicos e Científicos. 5ª edição, 2001.

[2] STEWART, James. *Cálculo*, Volume 1. Editora Cengage Learning, 7ª edição, 2013.